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Linear Differential–Algebraic Equations in Spaces of Integrable Functions

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The purpose of this note is the development of an L^2-theory for linear differential–algebraic equations

$$\tilde{T}x(t) := A(t)x(t)' + B(t)\,x(t) = f(t), \qquad t \in (a, b),\ x(t) \in R^m. \tag{1}$$

From the theory of differential–algebraic equations in spaces of continuous functions it is known that some kinds of such equations lead to ill-posed problems [4]. In order to solve these problems numerically we must use some regularization algorithms. A theory for (1) in Hilbert spaces gives us the possibility of exploiting the well-developed general theory of regularization methods.

In this note we try to answer the following questions:

(1) Which conditions have to be fulfilled such that $\tilde{T}$ is a closable operator? How is the domain of definition of the closed operator $T := \bar{\tilde{T}}$ characterized? If possible, give the representation of T.

(2) Let T be a closed operator now. It is well known that the generalized inverse T^+ exists. Furthermore, the domain of definition is $D(T^+) = R(T) + R(T)^\perp$ and T^+ is closed. T^+ is continuous if and only if the range $R(T)$ is closed, i.e., T is normally solvable [1]. Under which hypotheses is T normally solvable?

(3) Under which conditions is $R(T)$ not closed? In this case the differential–algebraic equation represents an essentially ill-posed problem.

The operators T with closed range describe the class of differential–algebraic equations which lead to well-posed problems if they are supplemented by appropriate boundary conditions. This class consists essentially of equations with constant index 1 (and 0, respetively) of the matrix pencil $(A(t), B(t))$ (for the definition see [2, 3]).

The class of essentially ill-posed problems contains differential–algebraic equations with higher index.

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At first we do not take into consideration any boundary conditions. It is well known that the form of the possible boundary conditions essentially depends on the matrices $A(t)$, $B(t)$. Therefore, we introduce boundary conditions at a later stage of our investigations. This procedure has no influence on the closedness of T and the closedness of $R(T)$, respectively.

Notations. In order to simplify the notations we denote the spaces $(L^2(a,b))^m$, $(H^1(a,b))^m$, etc. by L^2, H^1, etc. Let $(\cdot,\cdot)$ be the scalar product in L^2 and $\langle\cdot,\cdot\rangle$ the scalar product in R^m. If T is a linear operator, $D(T)$, $N(T)$, and $R(T)$ denote the domain, the kernel, and the range of T, respectively. T^* denotes the adjoint operator.

If X, Y are Banach spaces, let $B(X,Y)$ denote the Banach space of all continuous linear operators T with $D(T)=X$ and $R(T)\subseteq Y$. Moreover, let $B(X):=B(X,X)$.

Let $A\in L^\infty((a,b),B(R^m))$ be a matrix-valued function. Then we denote by A also the continuous linear operator $A\in B(L^2)$ defined by

$$(Ax)(t):=A(t)\,x(t).$$

Supposition 1. For (1), let

$$A\in W^{1,\infty}((a,b),B(R^m)),\qquad B\in L^\infty((a,b),B(R^m)). \tag{2}$$

Let $Q(t)$ and $R(t)$ be projectors onto $N(A(t))$ and $R(A(t))$, respectively. Let

$$P=I-Q,\qquad S=I-R. \tag{3}$$

Assume that

$$Q\in W^{1,\infty}((a,b),B(R^m)), \tag{4}$$

$$R\in W^{1,\infty}((a,b)\,B(R^m)). \tag{5}$$

LEMMA 1. *Let Supposition* 1 *hold. Then*

(i) $\dim N(A(t))=\text{const}$.

(ii) *For the Moore–Penrose generalized inverse* $A(t)^+$ *we have* $A(\cdot)^+\in W^{1,\infty}((a,b),B(R^m))$.

(iii) *Statements* (4), (5) *hold with* Q, R *replaced by the respective orthogonal projections* $Q^\perp$ *and* $R^\perp$.

(iv) *For the generalized inverse* $A(t)^+_{P(t),R(t)}$ *relative to the projectors* $P(t)$, $R(t)$ *it holds that* $A(\cdot)^+_{P(t),R(t)}\in W^{1,\infty}((a,b),B(R^m))$. (*For the definition see, e.g.,* [7].)

Proof. (i) is a consequence of (4). Together with (2) we get $A(\cdot)^+\in C([a,b],B(R^m))$ [6]. This is also true for the orthogonal projections $Q = I - A(\cdot)^+ A$ and $R = AA(\cdot)^+$. With [7, Formula (3.19)] we obtain

$$\begin{aligned} A(t+h)^+ - A(t)^+ = {} & -A(t+h)^+ (A(t+h) - A(t))\, A(t)^+ \\ & + A(t+h)^+\, A(t+h)^{+*}(A(t+h) - A(t))^*(I - R^\perp(t)) \\ & - Q^\perp(t+h)(A(t+h) - A(t))^* A(t)^{+*} A(t)^+. \end{aligned}$$

Since A is differentiable almost everywhere, this is true for $A(\cdot)^+$, too, and it holds that

$$\begin{aligned} \frac{d}{dt} A(t)^+ = {} & -A(t)^+ A'(t)\, A(t)^+ + A(t)^+ A(t)^{+*} A'(t)^* (I - R^\perp(t)) \\ & - Q^\perp(t)\, A'(t)^* A(t)^{+*} A(t)^+. \end{aligned}$$

Hence,

$$\left\| \frac{d}{dt} A(t)^+ \right\| \leqslant C \| A'(t) \|.$$

This proves (ii) and (iii). (iv) follows now from the representation [7]

$$A(\cdot)^+_{P(\cdot),\, R(\cdot)} = (I + P^\perp - P)\, A(\cdot)^+ (I - R^\perp + R). \quad \blacksquare$$

For the proof of the following theorem I am indebted to A. Neubauer.

THEOREM 1. *Let Supposition* 1 *hold. Let* $\tilde{T}$ *be defined by* (1) *with* $D(\tilde{T}) := C^\infty$. *Then* $\tilde{T}$ *is closable and*

$$D(\bar{\tilde{T}}) = H^1_P := \{ u \in L^2 \mid Pu \in H^1 \}. \tag{6}$$

Proof. It is well known that $\tilde{T}$ is closable iff $\tilde{T}^{**}$ exists. Moreover, $\bar{\tilde{T}} = \tilde{T}^{**}$. Partially integrating we obtain

$$\begin{aligned} (\tilde{T}x, y) & = \int_a^b \langle Ax' + Bx, y \rangle \, dt \\ & = \int_a^b \langle x, -(A^*y)' + B^*y \rangle \, dt + \langle x, A^*y \rangle \big|_a^b. \end{aligned}$$

Hence,

$$D(\tilde{T}^*) = \{ y \in L^2 \mid (A^*y) \in H^1,\ A^*y(a) = A^*y(b) = 0 \}.$$

If $A^*y \in H^1$, $(A^*)^+_{R^*, P^*} A^*y = R^*y \in H^1$ by Lemma 1(iv). Conversely, if

$R^*y \in H^1$, $A^*y = A^*R^*y \in H^1$. Here we used the fact that $A = RAP$ and $A^* = P^*A^*R^*$. Moreover, we have $A^*y = 0$ iff $R^*y = 0$. Hence,

$$D(\tilde{T}^*) = \{ y \in L^2 \mid R^*y \in H^1,\ R^*y(a) = R^*y(b) = 0 \}. \tag{7}$$

Since $D(\tilde{T}^*)$ is dense in L^2, $\bar{\tilde{T}} = \tilde{T}^{**}$ exists. Compute for $y \in D(\tilde{T}^*)$:

$$\begin{aligned}(x, \tilde{T}^*y) &= \int_a^b \langle x, -(A^*y)' + B^*y \rangle \, dt \\ &= \int_a^b \langle x, -(A')^*R^*y - A^*(R^*y)' + B^*y \rangle \, dt \\ &= \int_a^b \langle -RA'x + R(Ax)' + Bx, y \rangle \, dt + \langle Ax, R^*y \rangle \big|_a^b \\ &= \int_a^b \langle -RA'x + R(Ax)' + Bx, y \rangle \, dt.\end{aligned}$$

Hence,

$$D(\tilde{T}^{**}) = \{ x \in L^2 \mid Ax \in H^1 \}. \tag{8}$$

As above, $Ax \in H^1$ iff $Px \in H^1$. This proves the assertion. ∎

Because of Theorem 1 we obtain the following explicit representation of $T := \bar{\tilde{T}}$:

$$Tx = A(Px)' - AP'x + Bx, \qquad x \in D(T). \tag{9}$$

This representation suggests the use of the theory of abstract Sobolev spaces [8] in order to answer the latter two questions of the Introduction.

Our Sobolev space will be generated by the operator $Z_1 = Z = (P \cdot)'$ using the common methods. To be more precise, let $H^0 := L^2$, $H := C_0^\infty$. Let $Z\colon H \to H^0$ be given by $Zu := (Pu)'$, $u \in H$. Then

$$\begin{aligned}(Zu, v) &= \int_a^b \langle (Pu)', v \rangle \, dt = \int_a^b \langle Pu, v' \rangle \, dt = \int_a^b \langle u, -P^*v' \rangle \, dt \\ &= (u, -P^*v').\end{aligned}$$

Hence, $\hat{Z}\colon H \to H^0$ is defined by $\hat{Z}v = -P^*v'$. The maximal operator $Z_{\max}$ is given by

$$Z_{\max} = (\hat{Z})^*;$$

therefore,

$$u \in D(Z_{\max}) \text{ iff } \exists w \in H^0\ \forall v \in H\colon (v, w) = (-P^*v', u).$$

The Sobolev space $\tilde{H}_P^1$ is defined by

$$\tilde{H}_P^1 := D(Z_{\max})$$

equipped with the scalar product

$$(u, v)_P := (u, v) + (Z_{\max} u, Z_{\max} v).$$

LEMMA 2. *Let Supposition* 1 *hold. Then* $\tilde{H}_P^1 = H_P^1$. *Furthermore,* $Z_{\max} u = (Pu)'$ *for all* $u \in H_P^1$.

Proof. Let $u \in L^2$. Then $Pu \in H^1$ iff

$$\exists \bar{w} \in L^2\ \forall v \in H: (-v', Pu) = (v, \bar{w}). \tag{$*$}$$

Let $u \in D(Z_{\max})$. Now

$$\begin{aligned}(-v', Pu) &= \int_a^b \langle -v', Pu \rangle\, dt = \int_a^b \langle -P^* v', u \rangle\, dt = (-P^* v', u)\\ &= (v, w),\end{aligned}$$

that is, $(*)$ with $w = \bar{w}$.

Conversely, let $Pu \in H^1$. We have for all $v \in H$,

$$\begin{aligned}(-P^* v', u) &= \int_a^b \langle -P^* v', u \rangle\, dt = \int_a^b \langle -v', Pu \rangle\, dt = (-v', Pu)\\ &= (v, \bar{w}),\end{aligned}$$

i.e., $u \in D(Z_{\max})$ and $Z_{\max} u = (Pu)'$. ∎

When defining H_P^1 we used the projector P in the main. The next lemma shows that H_P^1 and its topology depend only on T, not on P.

LEMMA 3. *Let Supposition* 1 *hold. Let* $\bar{Q}(t)$ *be a projector onto* $N(A(t))$ *such that* $\bar{Q} \in W^{1,\infty}((a, b), B(R^m))$. *We set* $\bar{P} := I - \bar{Q}$. *Then* $H_P^1 = H_{\bar{P}}^1$ *and the norms are equivalent.*

Proof. Since $\bar{Q} = Q\bar{Q}$ and $Q = \bar{Q}Q$, we obtain $\bar{P} = \bar{P}P$ and $P = P\bar{P}$. Using (4) we obtain $H_{\bar{P}}^1 = H_P^1$. Furthermore,

$$\begin{aligned}\|u\|_P^2 &= \|u\|^2 + \|(Pu)'\|^2 = \|u\|^2 + \|(P\bar{P}u)'\|^2\\ &= \|u\|^2 + \|P'\bar{P}u + P(\bar{P}u)'\|^2 \leqslant C(\|u\|^2 + \|(\bar{P}u)'\|^2). \quad ∎\end{aligned}$$

In the following we will always assume that T is defined by (9) and that $D(T) = H_P^1$. Hence, $T \in B(H_P^1, H^0)$. Since T is closed in $H^0 = L^2$, T is

H^1_P-elliptic [8, Theorem II – 2.3(16)]. Therefore, there exists an $\alpha > 0$ such that

$$\|u\|^2_P \leqslant \|Tu\|^2 + \|u\|^2 \tag{10}$$

for all $u \in H^1_P$.

Normally solvable operators are characterized by the following proposition:

PROPOSITION. *Let* $T \in B(X, Y), X, Y, Z$ *be Hilbert spaces with* $\dim N(T) \leqslant \dim Z$. *Then T is normally solvable and* $\dim N(T) < \infty$ *iff there exists a compact operator* $K \in B(X, Z)$ *and a* $k_0 > 0$ *such that*

$$\|u\| \leqslant k_0(\|Tu\| + \|Ku\|) \tag{11}$$

for all $u \in X$.

In the theory of ordinary differential operators we can use the fact that the imbedding $H^1 \to L^2$ is completely continuous (condition (R) of [8]). In this case, (11) is an immediate consequence of (10). But in our situation the condition (R) is not fulfilled in general, i.e., the imbedding $H^1_P \to L^2$ is not completely continuous. Our aim is the specification of conditions which allow the construction of a suitable operator K for (11) to hold. The theory of differential–algebraic equations in spaces of continuous functions suggests that this is possible in the following two cases (cf. [3]):

(I) A is non-singular almost everywhere.

(II) The equation is transferable, i.e., the pencil $(A(t), B(t))$ is regular and $\operatorname{ind}(A(t), B(t)) = 1$ almost everywhere.

Case I is trivial and of less interest:

LEMMA 4. *Let Supposition* 1 *hold. Let A be non-singular almost everywhere and let $V \subseteq H^1_P$ be a closed subspace. Then $T|_V$ is normally solvable and* $\dim N(T|_V) < \infty$.

Proof. Because of $N(A(t)) = \{0\}$ almost everywhere we have $Q = 0$, i.e., $P = I$. Hence, $H^1_P = H^1$. Now, (10) implies (11) with $K = J$, where J is the imbedding of H^1 into L^2.

Remark. If the suppositions of Lemma 4 hold, (1) is a system of ordinary differential equations. Lemma 1(iv) gives $A(\cdot)^{-1} \in L^\infty((a, b), B(R^m))$. This is the well-known ellipticity condition for ordinary differential operators.

LEMMA 5. *Let Supposition* 1 *hold. Let the pencil* $(A(t), B(t))$ *be regular* *and* $\operatorname{ind}(A(t), B(t)) = 1$ *for almost every* $t \in (a, b)$. *Then,* $G(t) := S(t)\, B(t)|_{N(A(t))}\colon N(A(t)) \to R(S(t))$ *is bijective for almost every* $t \in (a, b)$. *Suppose* $G(\cdot)^{-1} \in L^\infty$.

If $V \subseteq H_P^1$ *is closed, then* $T|_V$ *is normally solvable and* $\dim N(T|_V) < \infty$.

Proof. Using

$$\|x\|^2 = \|Px + Qx\|^2 \leqslant 2(\|Px\|^2 + \|Qx\|^2)$$

we obtain by (10)

$$\|x\|_P^2 \leqslant \|Tx\|^2 + \|x\|^2 \leqslant \|Tx\|^2 + 2\,\|Px\|^2 + 2\,\|Qx\|^2. \tag{$**$}$$

Multiplying (9) by S gives

$$STx = SBx = SB(Px + Qx)$$

and

$$SBQx = S(Tx - BPx).$$

Hence,

$$Qx = G^{-1}S(Tx - BPx)$$

which implies

$$\|Qx\| \leqslant \|G^{-1}\|_\infty \|S\|_\infty (\|Tx\| + \|B\|_\infty \|Px\|)$$

and

$$\|Qx\|^2 \leqslant C(\|Tx\|^2 + \|Px\|^2).$$

Using $(**)$ we obtain

$$\|x\|_P^2 \leqslant k(\|Tx\|^2 + \|Px\|^2),$$

where $k = 2(C+1)/\alpha$.

Now it is sufficient to show that the mapping $x \in H_P^1 \mapsto Px \in L^2$ is completely continuous. If $M \subseteq H_P^1$ is bounded, then

$$\begin{aligned}\|Px\|^2 + \|(Px)'\|^2 &\leqslant \|P\|_\infty^2\, \|x\|^2 + \|(Px)'\|^2\\ &\leqslant \|P\|_\infty^2\, \|x\|_P^2\end{aligned}$$

for all $x \in M$. Therefore, $P(M)$ is bounded in H^1; hence $P(M) \subseteq L^2$ is precompact. ∎

THEOREM 2. *Let Supposition 1 hold. Let $H := A + SBQ$ be bijective for almost every $t \in (a, b)$ and $H(\cdot)^{-1} \in L^{\infty}((a, b), B(R^m))$.*

If $V \subseteq H_P^1$ is closed, then $T|_V$ is normally solvable and $\dim N(T|_V) < \infty$.

Proof. It is easy to show that the bijectivity of $H(t)$ is equivalent to the regularity of the pencil $(A(t), B(t))$ and that $\operatorname{ind}(A(t), B(t)) = 1$ [3, Appendix A]. Furthermore,

$$H(t)|_{N(A(t))} = S(t)\, B(t)|_{N(A(t))} = G(t).$$

Hence,

$$G(t)^{-1} = H(t)^{-1}|_{R(S(t))},$$

which implies $\|G^{-1}\|_{\infty} \leqslant \|H^{-1}\|_{\infty}$. Lemma 5 gives the assertion. ∎

Remark. The essential condition for T to be normally solvable is given by Lemma 5. But the condition on G can usually not be verified. It is more convenient to use the matrix function H.

Now we switch over to the third question. The answer is considerably more complicated than the previous ones. Therefore, next we consider a special case. The main idea of the proof consists of the fact that problems with higher nilpotency include differentiation problems. Hence, such problems have solutions for smooth right-hand sides, only. But sets of differentiable functions are not closed in L^2.

LEMMA 6. *We consider* (9) *with $B(t) = I$. Let Supposition* 1 *hold. Suppose that there is a subinterval $J \subseteq (a, b)$ such that* $\operatorname{ind} A(t) = k \geqslant 2$ *almost everywhere in J. Let $M := A^{k-1}(I - AA^{\mathrm{D}}) \in W^{1,\infty}(J, B(R^m))$ where A^{D} denotes the Drazin generalized inverse of A. Let $V \subseteq H_P^1$ be closed.*

If V is sufficiently large, $R(T|_V)$ is not closed. More precisely, if $\dim M(V)|_J = \infty$, *then $R(T|_V)$ is not closed.*

Proof. Since $\operatorname{ind} A(t) = k$ almost everywhere on J, $M(t) \neq 0$ holds on J. Moreover,

$$AM = A^k(I - AA^{\mathrm{D}}) = A^k - A^k = 0, \qquad \text{a.e. on } J.$$

Because of $Px \in H^1$ it holds that $Px|_J \in (H^1(J))^m$. Hence, using properties of A^{D} we obtain

$$\begin{aligned} MTx &= MA(Px)' - MAP'x + Mx \\ &= Mx \\ &= (I - AA^{\mathrm{D}})\, A^{k-1}x \\ &= (I - AA^{\mathrm{D}})\, A^{k-1}Px \\ &= MPx \in (H^1(J))^m. \end{aligned} \tag{***}$$

Suppose $R(T)$ to be closed. This is true for $R(T|_J)$ in $(L^2(J))^m$, too. Since $(I-AA^D)$ is a projection, $R((I-AA^D)\,T|_J)$ is closed. By Lemma 1(ii) $A^+\in L^\infty((a,b), B(R^m))$, and hence $R(MT|_J)$ is closed. But it holds that

$$R(MT|_J)=R(M|_J)=M(V)|_J.$$

We obtain dim $M(V)|_J<\infty$ by (***) using the fact that a linear subset $W\subseteq (H^1(J))^m$ is closed in $(L^2(J))^m$ iff dim $W<\infty$. Now the proposition follows. ∎

Remark. It is clear that the closedness of $R(T)$ depends essentially on V. Our aim is the consideration of boundary-value problems for differential–algebraic equations, i.e., V is defined by some kind of boundary conditions. For the time being, let

$$V:=\{x\in H_P^1 \mid D_a x(a)+D_b x(b)=0\} \tag{12}$$

with some matrices $D_a, D_b\in B(R^m, R^{m'})$. The next theorem shows that this leads to a non-closed range of T, i.e., essentially ill-posed problems.

THEOREM 3. *Let the suppositions of Lemma 6 be true and let V be defined by* (12). *Setting $N(t):=N(A(t)^{k-1})$ suppose that there is a function $z\in(H^1(J))^m$ with $z(t)\in N(t)\cap R(A(t))$, $z(t)\notin N(A(t)^{k-2})$ a.e. on J. Then $R(T)$ is not closed.*

Proof. For convenience we choose Q to be the orthoprojection. Analogous to the proof of [5, Lemma 4] we obtain.

$$N(A^k)=N\oplus A^+(N\cap R(A)) \qquad \text{a.e. on } J.$$

Let z be as above. Define

$$W:=\{u\in L^2 \mid \exists\varphi\in H^1(J)\colon u(t)=\varphi(t)A(t)^+z(t) \text{ a.e. on } J,$$
$$u|_{(a,b)\setminus J}=0\}.$$

Hence, for all $u\in W$,

$$Pu|_J=\varphi PA^+z|_J=\varphi A^+z|_J.$$

By Lemma 1(ii), $W\subseteq H_P^1$. Furthermore, $W\subseteq V$. Moreover,

$$\begin{aligned}M(\varphi A^+z)&=\varphi A^{k-1}(I-AA^D)\,A^+z\\&=\varphi A^{k-2}(I-AA^D)\,z\\&=\varphi A^{k-2}z\end{aligned}$$

almost everywhere on J. Thus, $M(W)|_J$ is algebraically isomorphic to $H^1(J)$, which proves $\dim M(V)|_J \geqslant \dim M(W)|_J = \infty$. ∎

Let us now return to general regular matrix pencils $(A(t), B(t))$. Then, an $\alpha(t)$ exists for almost every $t \in (a, b)$ such that $\alpha(t)\, A(t) + B(t)$ is bijective. Suppose that there is an $\alpha \in L^\infty$ such that $(\alpha A + B)^{-1} \in W^{1,\infty}((a, b), B(R^m))$. Let $E\colon L^2 \to L^2$ and $F\colon H_P^1 \to H_P^1$ be defined by

$$Ef(t) := (\alpha(t)\, A(t) + B(t))^{-1} f(t),$$

$$Fy(t) := \exp\left(\int_a^t \alpha(s)\, ds\right) y(t).$$

Then both mappings are continuous bijections. Define $\hat{T} := ETF$. An easy calculation gives

$$\hat{T}y = \hat{A}y' + y,$$

where $\hat{A} = EAF$. Moreover, $N(\hat{A}(t)) = N(A(t))$. Hence, $\hat{T}$ fulfils Supposition 1. Furthermore, the matrix pencil $(A(t), I)$ is regular and its index is equal to that of $(A(t), B(t))$ [3, Corollary 1.3.2.9]. Since the problem of solving $Tx = f$ is ill-posed if and only if $\hat{T}y = g$ is so, we can apply Theorem 3 resp. Lemma 6 to obtain conditions for the ill-posedness of $Tx = f$.

In the next stage of our considerations we introduce boundary conditions. In particular, we want to characterize the conditions which guarantee that T is Fredholm of index zero. In accordance with Theorems 2 and 3 we only consider the case $\operatorname{ind}(A(t), B(t)) = 1$. First, from Theorem 2 it follows that T is semi-Fredholm on every closed subspace $V \subseteq H_P^1$. The subspaces V will be defined by boundary conditions. In contrast to (12) we define V by means of continuous linear functionals. Let $V \subseteq H_P^1$ be closed and $W := \{l \in H_P^{1*} \mid l(x) = 0\ \forall x \in V\}$. Then $V = \{x \in H_P^1 \mid l(x) = 0\ \forall l \in W\}$. We choose W to be a subset of $U := \operatorname{lin}\{\delta_a^i, \delta_b^i \mid 1 \leqslant i \leqslant m\}$ where $\delta_t^i(x) := x^i(t)$.

LEMMA 7. *Let Supposition* 1 *hold and* $l \in U$, *Then* $l \in H_P^{1*}$ *if and only if* $l(x) = l(Px)$ *for all* $x \in H^1$.

Proof. Let $l = \langle c, \delta_a \rangle + \langle d, \delta_b \rangle$ with c, $d \in R^m$, $\delta_t = (\delta_t^1, ..., \delta_t^m)^T$, $t = a, b$.

$$(\leftarrow)\quad \begin{aligned} l(x) &= \langle c, \delta_a x \rangle + \langle d, \delta_b x \rangle \\ &= \langle c, \delta_a Px \rangle + \langle d, \delta_b Px \rangle \\ &= \langle c, (Px)(a) \rangle + \langle d, (Px)(b) \rangle. \end{aligned}$$

Because of $Px \in H^1$ the mappings $x \mapsto (Px)(a)$ and $x \mapsto (Px)(b)$ are continuous. Hence, $l \in H_P^{1*}$.

($\rightarrow$) Let $l \in H_P^{1*}$. Then

$$l(x) = l(Px + Qx) = l(Px) + l(Qx).$$

Assume $l \circ Q \neq 0$. Let $x \in H_P^1$ such that $l(Qx) = \alpha \neq 0$. Now define

$$y(t) := \frac{x(b) - x(a)}{b - a}(t - a) + x(a), \qquad z := Qy.$$

Hence, $z \in H_P^1 \cap C$, $Pz = 0$, $z(a) = (Qx)(a)$, $z(b) = (Qx)(b)$. Thus, $l(z) = l(Qx) = \alpha$. Let $\alpha_n \in H^1(a, b)$ be defined by

$$\alpha_n(t) := \begin{cases} -n^2(t-a) + n, & a \leqslant t \leqslant a + \dfrac{1}{n} \\ 0, & a + \dfrac{1}{n} < t < b - \dfrac{1}{n}, \\ -n^2(t-b) + n, & b - \dfrac{1}{n} \leqslant t \leqslant b. \end{cases}$$

Let $z_n := \alpha_n z \in H_P^1$. Now $\|z_n\|_P \leqslant \|z\|_C \, \|\alpha_n\| \leqslant k$, but $l(z_n) = n\alpha \to \infty$ in contradiction to $l \in H_P^{1*}$. ∎

Let T be defined by (9) and let $D(T) = V \subseteq H_P^1$ be closed, where V is given by boundary conditions:

$$\begin{aligned} &V := \{x \in H_P^1 \mid l(x) = 0 \ \forall l \in W\}, \\ &W \subseteq U \cap H_P^{1*}. \end{aligned} \tag{13}$$

Hence, for all $y \in C_0^\infty$,

$$\begin{aligned} (Tx, y) &= \int_a^b \langle A(Px)' - AP'x + Bx, y \rangle \, dt \\ &= \int_a^b \langle x, -(A^*R^*y)' + (B - AP')^* y \rangle \, dt. \end{aligned}$$

Thus, $T^*|_{C_0^\infty}$ has the representation

$$T^*y = -A^*(R^*y)' + (B - RA' - AP')^* y. \tag{14}$$

Therefore, we have the following lemmas.

LEMMA 8. *Let Supposition* 1 *hold. Let T be given by* (9) *with $D(T) = V$ defined by* (13). *Then*:

(i) *S^* and R^* are projections onto $N(A^*)$ and $R(A^*)$, respectively.*

(ii) *$D(T^*) := V_* \subseteq H^1_{R^*}$ is closed. T^* is V_*-elliptic.* ∎

LEMMA 9. *Let Supposition* 1 *hold and let $A(t)$ be regular almost everywhere on (a, b). Then, T^* is normally solvable and* $\dim N(T^*) < \infty$.

When investigating the case of an index-1-pencil the following lemma is useful.

LEMMA 10. *Let Supposition* 1 *hold. Set $H = A + SBQ$ and $H_* = -A^* + Q^*(B - RA' - AP' - R'A)^*S^*$. Then $H_* = H^*$.*

Proof. Let $t \in (a, b)$ be fixed and $x, y \in R^m$. Then

$$\begin{aligned} \langle Hx, y\rangle &= \langle (A + SBQ)\,x, y\rangle \\ &= \langle (A + SBQ)\,x - S(RA' + AP' + R'A)\,Qx, y\rangle \\ &= \langle x, H_*y\rangle. \quad \blacksquare \end{aligned}$$

THEOREM 4. *Let the suppositions of Theorem* 2 *be true. Then T^* is normally solvable and* $\dim N(T^*) < \infty$.

Proof. Comparing the representations (14) and (9) we conclude that the relevant matrix H for T^* is given by $-A^* + Q^*(B - RA' - AP' - R'A)^*S^*$. Now, Lemma 10 and Theorem 2 prove the assertion. ∎

Theorem 4 shows that T is semi-Fredholm with finite index. Now we are interested in conditions which guarantee that $\dim N(T) = \dim N(T^*)$.

We introduce the following notations:

$$U_P := U \cap H_P^{1*}, \qquad U_{R^*} := U \cap H_{R^*}^{1*},$$

$$\dim R(A(a)) = \dim R(A(b)) =: m',$$

$$\text{i.e., } \dim N(A(a)) = \dim N(A(b)) = m - m'.$$

We have

$$(Tx, y) = (x, T^*y) + [x, y]_a^b,$$

$$[x, y]_a^b := \langle APx, R^*y\rangle\big|_a^b$$

for all $x \in H_P^1$, $y \in H_{R^*}^1$.

LEMMA 11. *Let Supposition* 1 *hold. If* $\{l_1, ..., l_{2m'}\}$ *is a basis of* U_P, *then there exists a basis* $\{\bar{l}_1, ..., \bar{l}_{2m'}\}$ *of* U_{R^*} *such that*

$$[x, y]_a^b = \sum_{i=1}^{2m'} l_i(x)\, \bar{l}_i(y)$$

for all $x \in H_P^1$ *and* $y \in H_{R^*}^1$.

Proof. By Lemma 3, $U_P = \operatorname{lin}\{\delta_a^1 P, ..., \delta_a^m P, \delta_b^1 P, ..., \delta_b^m P\}$. Therefore, $\dim U_P = \dim R(P(a)) \times R(P(b)) = \dim N(A(a)) \times N(A(b)) = 2m'$. Analogously, $\dim U_{R^*} = 2m'$. Let

$$\xi := \begin{pmatrix} \delta_a P \\ \delta_b P \end{pmatrix}, \qquad \eta := \begin{pmatrix} \delta_a R^* \\ \delta_b R^* \end{pmatrix}, \qquad M := \begin{pmatrix} -A(a) & 0 \\ 0 & A(b) \end{pmatrix},$$

$$L := (l_1, ..., l_{2m'})^T.$$

Moreover, let $\tilde{L} := (\tilde{l}_1, ..., \tilde{l}_{2m'})^T$ be an arbitrary basis of U_{R^*}. Then

$$[x, y]_a^b = \eta^T(y)\, M \xi(x).$$

Furthermore, there are matrices $T_1, T_2 \in B(R^{2m'}, R^{2m})$ such that $\xi = T_1 L$, $\eta = T_2 \tilde{L}$. Hence, $[x, y]_a^b = \tilde{L}(y)^T T_2^T M T_1 L(x)$. Let $\bar{L} := T_1^T M T_2 \tilde{L}$. Then $[x, y]_a^b = \bar{L}(y)^T L(x)$. It is now sufficient to show that $\bar{L}$ is a basis of U_{R^*}. Because of $\tilde{L} \subseteq U_{R^*}$, $\tilde{L}(y) = \tilde{L}(R^* y)$; thus $\bar{L}(y) = \bar{L}(R^* y)$, i.e., $\bar{L} \subseteq U_{R^*}$. In order to show the linear independence of the elements of $\bar{L}$ we prove the bijectivity of $T_2^T M T_1 \in B(R^{2m'})$. Compute

$$\begin{aligned} \begin{pmatrix} R(a)^* & 0 \\ 0 & R(b)^* \end{pmatrix} T_2 \tilde{L}(y) &= \begin{pmatrix} R(a)^* & 0 \\ 0 & R(b)^* \end{pmatrix} \begin{pmatrix} R(a)^* y(a) \\ R(b)^* y(b) \end{pmatrix} \\ &= \begin{pmatrix} R(a)^* y(a) \\ R(b)^* y(b) \end{pmatrix} = T_2 \tilde{L}(y) \end{aligned}$$

for all $y \in H_{R^*}^1$. Since the components of $\tilde{L}$ are linearly independent, $R(\tilde{L}) = R^{2m'}$. Thus,

$$R(T_2) \subseteq R\left(\begin{pmatrix} R(a)^* & 0 \\ 0 & R(b)^* \end{pmatrix}\right) = R(R(a)^*) \times R(R(b)^*).$$

Moreover, $\dim R(T_2) = \dim \operatorname{lin}\{\delta_a^1 R^*, ..., \delta_b^m R^*\} = \dim U_{R^*} = 2m'$. Hence, $N(T_2) = \{0\}$ and $R(T_2) = R(R(a)^*) \times R(R(b)^*)$. Analogously, $N(T_1) = \{0\}$ and $R(T_1) = R(P(a)) \times R(P(b))$. Moreover, $N(M) = N(A(a)) \times N(A(b))$, $R(M) = R(A(a)) \times R(A(b))$.

Now, let $z \in R^{2m'}$, $T_2^T M T_1 z = 0$. Hence, $MT_1 z \in N(T_2^T) \cap R(M)$. It holds that

$$\begin{aligned} N(T_2^T) \cap R(M) &= R(T_2)^{\perp} \cap R(A(a)) \times R(A(b)) \\ &= [R(R(a)^*) \times R(R(b)^*)]^{\perp} \cap R(R(a)) \times R(R(b)) \\ &= N(R(a)) \times N(R(b)) \cap R(R(a)) \times R(R(b)) \\ &= \{0\}. \end{aligned}$$

Thus, $MT_1 z = 0$, i.e., $T_1 z \in N(M) \cap R(T_1)$,

$$\begin{aligned} N(M) \cap R(T_1) &= N(A(a)) \times N(A(b)) \cap R(P(a)) \times R(P(b)) \\ &= R(I - P(a)) \times R(I - P(b)) \cap R(P(a)) \times R(P(b)) \\ &= \{0\}. \end{aligned}$$

Therefore, $T_1 z = 0$. Since $N(T_1) = \{0\}$, $z = 0$. This means that $T_2^T M T_1$ is injective and consequently bijective. ▮

LEMMA 12. *Let the suppositions of Theorem 2 be true. The homogeneous equation $Tx = 0$ has exactly m' linear independent solutions in H_P^1.*

Proof. Let $x \in H_P^1$. Then

$$\begin{aligned} & Tx = 0 \\ \Leftrightarrow & \begin{cases} RA(Px)' - RAP'x + RBx = 0 \\ SBx = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} A_{P,R}^+ A(Px)' - A_{P,R}^+ AP'x + A_{P,R}^+ RBx = 0 \\ SB(Px + Qx) = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} P(Px)' - PP'x + A_{P,R}^+ RBx = 0 \\ SBQx = -SBPx \end{cases} \\ \Leftrightarrow & \begin{cases} (Px)' - P'Px - (PP' - A_{P,R}^+ RB)\, x = 0 \\ Qx = -H^{-1} SBPx. \end{cases} \qquad (****) \end{aligned}$$

(a) We show the following proposition: Let $y \in H^1$ be a solution of

$$y' - P'y - (PP' - A_{P,R}^+ RB)(y - H^{-1} SBy) = 0. \qquad (*****)$$

Then $Qy = 0$ iff $Qy(t_0) = 0$ for some $t_0 \in [a, b]$.

$(\rightarrow)$ Obvious.

($\leftarrow$) Multiply (*****) by Q. Regarding $QP=0$, $QA^+_{P,R}=0$ we obtain

$$\begin{aligned}0 &= Qy' - QP'y\\ &= (Qy)' - Q'y + QQ'y\\ &= (Qy)' - Q'Qy.\end{aligned}$$

Now $Qy \in H^1$, $Qy(t_0)=0$ imply $Qy=0$.

(b) Equation (****) is an ordinary differential equation for Px and a finite-dimensional assignment for Qx. This equation has exactly m linearly independent solutions $y_1, ..., y_m \in H^1$ with Px replaced by y. Let

$$y = \sum_{i=1}^{m} \alpha_i y_i.$$

Then $y \in R(P)$ iff $Py = y$, i.e.,

$$\sum_{i=1}^{m} \alpha_i (I-P)\, y_i = 0.$$

By (a), this is true iff

$$\sum_{i=1}^{m} \alpha_i (I - P(a))\, y_i(a) = 0.$$

The linear system of equations has exactly m–rank$(I-P(a)) = m'$ lineaarly independent solutions. The proposition follows, since the solutions of (****) are linearly independent iff they are so for $t=a$. ∎

THEOREM 5. *Let the suppositions of Theorem 2 be true. Let* $W \subseteq U_P$, dim $W=n$ $(0 \leqslant n \leqslant 2m')$, $D(T) := V = \{x \in H^1_P \mid l(x) = 0\ \forall l \in W\}$, *and let* $\{l_1, ..., l_n\}$ *be a basis of* W.

Then there exist linearly independent functionals $\bar{l}_{n+1}, ..., \bar{l}_{2m'} \in U_{R^*}$ *such that*

$$\begin{aligned}T^*y &= -A^*(R^*y)' + (B - RA' - AP')^*\, y,\\ D(T^*) &= V_* := \{y \in H^1_{R^*} \mid \bar{l}_j(y) = 0, j = n+1, ..., 2m'\}.\end{aligned}$$

Moreover, dim $N(T^*) = n - m' + \dim N(T)$.

Proof. Choose $l_{n+1}, ..., l_{2m'}$ such that $\{l_1, ..., l_{2m'}\}$ is a basis of U_P. Select $\bar{l}_1, ..., \bar{l}_{2m'} \in U_{R^*}$ according to Lemma 11. Statements (14) and (15) imply

$$D(T^*) = \{y \in H^1_{R^*} \mid \bar{l}_j(y) = 0, j = n+1, ..., 2m'\}.$$

Let $z_1, ..., z_{m'}$ be a fundamental system of $T^*z=0$, $z \in H^1_{R^*}$ (cf. Lemma 12). Let

$$z=\sum_{i=1}^{m'} \alpha_i z_i.$$

$z \in V_*$ iff

$$\sum_{i=1}^{m'} \alpha_i \bar{l}_j(z_i)=0, \qquad j=n+1, ..., 2m'.$$

Define

$$\bar{L} := \begin{pmatrix} \bar{l}_{n+1}(z_1) & \cdots & \bar{l}_{n+1}(z_{m'}) \\ \vdots & & \vdots \\ \bar{l}_{2m'}(z_1) & \vdots & \bar{l}_{2m'}(z_{m'}) \end{pmatrix}, \qquad k' := \text{rank } \bar{L}.$$

Therefore, there exist exactly $m'-k'$ linearly independent solutions of the system $\bar{L}\alpha=0$, i.e., $r' := \dim N(T^*) = m'-k'$. Analogously, let $y_1, ..., y_{m'}$ be a fundamental system of $Tx=0$, $y \in H^1_P$,

$$L := \begin{pmatrix} l_1(y_1) & \cdots & l_1(y_{m'}) \\ \vdots & & \vdots \\ l_n(y_1) & \cdots & l_n(y_{m'}) \end{pmatrix}, \qquad k := \text{rank } L.$$

Hence, $r := \dim N(T) = m'-k$.

Without loss of generality let $y_1, ..., y_{m'-k}$ be linearly independent. Consider the linear system of equations $\bar{L}^T\lambda=0$, $\lambda \in R^{2m'-n}$. Since rank $\bar{L}=$ rank $\bar{L}^T$ there are at most $2m'-n-k'$ linearly independent solutions. Conversely, there are at least $m'-k$ linearly independent solutions, namely $\lambda^j := (l_{n+1}(y_j), ..., l_{2m'}(y_j))^T, j=1, ..., m'-k$. Therefore, $m'-k \leqslant 2m'-n-k'$, i.e., $k' \leqslant m'-n+k$. Replacing the equation by its adjoint gives $k' \geqslant m'-n+k$. Hence, $r'=n-m'+r$.

COROLLARY. *Under the hypotheses of Theorem* 5 *T, $D(T)=V$, is Fredholm of index $n-m'$. Especially, if $n=m'$, injectivity and surjectivity of T are equivalent.*

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